Length and Area of Circles

Pi in the Sky

Lesson 32-1 Circumference and Area of a Circle

My Notes

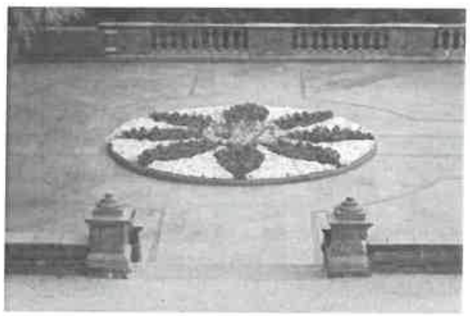
Circumference
$C = \pi d$ or $2\pi r$

Learning Targets:

- Develop and apply a formula for the circumference of a circle.
- Develop and apply a formula for the area of a circle.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Identify a Subtask

Lance owns The Flyright Company. His company specializes in making parachutes and skydiving equipment. After returning from a tour of Timberlake Gardens, he was inspired to create a garden in the circular drive in front of his office building. Lance plans to hire a landscape architect to design his garden. The architect needs Lance to provide the area and circumference of the proposed garden so the architect can estimate a budget.



Lance relies on the measures of some familiar objects to help him understand the measures of a circle.

Object	Distance Around the Object, C	Diameter, d	$\frac{C}{d}$
Coin	7.85 cm	2.5 cm	3.14
Clock	40.84 mm	13 mm	3.14
Frisbee	62.83 in.	20 in.	3.14

1. Complete the table to find the ratio of $C : d$.
2. Make a conjecture about the relationship between the circumference and the diameter of circular objects.

The ratio of the circumference to the diameter for any circular object is about 3.14.

The ***circumference*** of a circle is the distance around the circle. Circumference is measured in linear units. The ratio of the circumference to the diameter of any circle is designated by the Greek letter π (pi).

3. Write a formula, in terms of diameter, d, that can be used to determine the circumference of a circle, C.

$C = \pi d$

4. Write a formula, in terms of the radius, r, that can be used to determine the circumference of a circle, C.

$C = 2\pi r$

5. What information does the circumference provide about the garden Lance wants to create?

The circumference is the distance around the garden

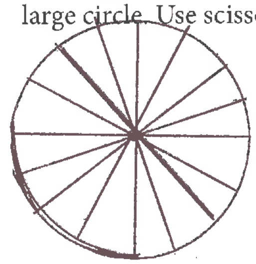
CONNECT TO HISTORY

The first known calculation of π (pi) was done by Archimedes of Syracuse (287–212 BCE), one of the greatest mathematicians of the ancient world. Archimedes approximated the area of a circle using the Pythagorean Theorem to find the areas of two regular polygons: the polygon *inscribed within the circle* and the polygon *within which the circle was circumscribed*. Since the actual area of the circle lies between the areas of the inscribed and circumscribed polygons, the areas of the polygons gave upper and lower bounds for the area of the circle. Archimedes knew that he had not found the value of pi but only an approximation within those limits. In this way, Archimedes showed that pi is between $3\frac{10}{71}$ and $3\frac{1}{7}$.

My Notes

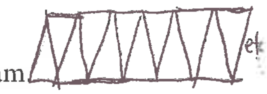
Lance uses a compass and paper folding to develop the formula for the area of a circle.

6. Use appropriate tools strategically. Use a compass to draw a large circle. Use scissors to cut the circle out.



7. Divide the circle into 16 equal ***sectors***.

8. Cut the sectors out and arrange the 16 pieces to form a parallelogram.



9. Write the formula for the area of a parallelogram, in terms of base *b* and height *h*.

$A = bh$

MATH TERMS

A **sector** is a pie-shaped part of a circle. A sector is formed by two radii and the arc determined by the radii.

10. Explain how the height of the parallelogram is related to the radius of the circle.

The height of the parallelogram is equal to the radius of the circle, so $h=r$.

11. Explain how the base of the parallelogram is related to the circumference of the circle.

The base of the parallelogram is equal to one-half of the circumference of the circle, so $b=\frac{1}{2}C$

12. How does the area of the parallelogram compare to the area of the original circle?

The areas are the same.

13. Construct viable arguments. Use your knowledge of area of parallelograms to prove the area formula of a circle.

$A_{parallelogram} = bh = \pi r(r) = \pi r^2 = A_{circle}$

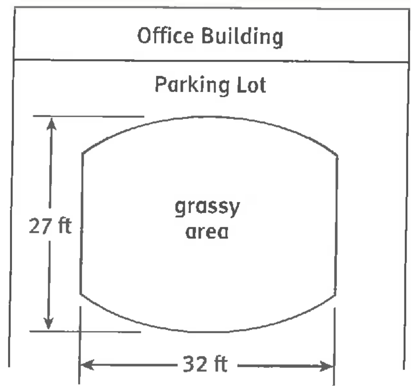
14. What information does the area provide about the garden Lance wants to create?

The area is the amount of ground the garden covers.

My Notes

Area
$A = \pi r^2$

The layout of The Flyright Company's building and parking lot is shown below.



The dimensions of the grassy area of the parking lot are 32 ft by 27 ft.

15. What are the radius, circumference, and area of the largest circle that will fit in the grassy area? Justify your answer.

$r = \frac{27}{2} = 13.5$ ft; $C = 27\pi$ ft ≈ 84.823 ft
$A = \pi(13.5)^2 = 182.25\pi$ ft^2 ≈ 572.555 ft^2

DISCUSSION GROUP TIP

As you share ideas with your group, be sure to explain your thoughts using precise language and specific details to help group members understand your ideas and reasoning.

Even though larger circular gardens will fit in the grassy area, Lance decides that he would like the garden to have a diameter of 20 feet.

16. The landscape architect recommends surrounding the circular garden with decorative edging. The edging is sold in 12-foot sections that can bend into curves. How many sections of the edging will need to be purchased to surround the garden? Justify your answer.

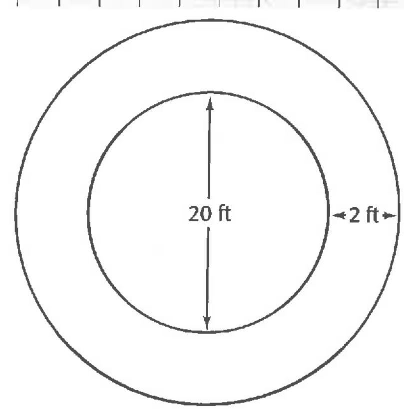
Lance must purchase six sections
$C = 20\pi \approx 62.823$ ft
$62.832 \div 12 = 5.236$ (Round up to 6 acres.)

17. To begin building the garden, the landscape architect needs to purchase soil. To maintain a depth of one foot throughout the garden, each bag can cover 3.5 square feet of the circle. How many bags of soil need to be purchased? Show the calculations that lead to your answer.

Lance must purchase 90 bags.
$A = \pi(10)^2 \approx 314$ ft^2; $314 \div 3.5 \approx 89.7$

18. Lance and the architect are discussing the possibility of installing a sidewalk around the outside of the garden, as shown in the diagram. Determine the area of the sidewalk.

44π ft^2 ≈ 138.23 ft^2



19. Critique the reasoning of others. Darnell claims that the circumference of a circular garden with radius 12 meters is greater than the perimeter of a square garden with side 18 meters. Do you agree? Explain your reasoning.

Yes; the circumference is 75.36 meters, and the perimeter of the square is 72 meters.

My Notes

20) The area of a circle is the number of units needed to completely cover its surface, while the circumference is a measure of distance.

21) $r = \frac{C}{2\pi}$

22) $A = \frac{C^2}{4\pi}$

23) The area will decrease.

24) $A = \pi(9)^2 = 81\pi \text{ ft}^2 \approx 254.5 \text{ ft}^2$

25) 10π cm or 31.42 cm

26) $1.5 \text{ ft} \cdot \pi \cdot 25 = 37.5\pi \text{ ft}$ or 117.8 ft

27) Yes, when the radius is 2.

$A = C$

$\pi r^2 = 2\pi r$

$r^2 = 2r$

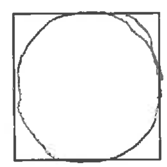
$r = 2$

Check Your Understanding

20. Explain why the units of measure for the circumference of a circle are linear, and those of the area of a circle are in square units.
21. Express the radius of a circle in terms of its circumference.
22. Express the area of a circle in terms of its circumference.
23. If Lance decides to increase the width of the walkway to 3 feet and keep the outer radius at 12 feet, what will happen to the area of the garden?

LESSON 32-1 PRACTICE

24. A circular rotating sprinkler sprays water over a distance of 9 feet. What is the area of the circular region covered by the sprinkler?
25. **Reason quantitatively.** What is the greatest circumference of the circle that can be inscribed in the square below?



10 cm

26. Suppose a landscaper pushes a wheelbarrow so that the wheel averages 25 revolutions per minute. If the wheel has a diameter of 18 inches, how many feet does the wheelbarrow travel each minute?
27. **Construct viable arguments.** Can the area of a circle ever equal the circumference of a circle? Support your answer with proof.

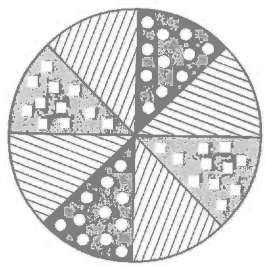
My Notes

Learning Targets:

- Develop and apply a formula for the area of a sector.
- Develop and apply a formula for arc length.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Create Representations, Quickwrite

Lance told the landscape architect that he would like the garden to resemble a colorful parachute with different flowers in alternating areas of the garden. The landscaper developed the sketch shown below.



1. **Model with mathematics.** The circular garden is divided into 8 equal parts.
   a. What portion of the total area of the circle is each part?

   $\frac{1}{8}$

   b. Recall that Lance would like the diameter of the circular garden to be 20 feet. Determine the area of each sector of the garden. Show the calculations that led to your answer.

   $A = \frac{\pi(10)^2}{8} = 12.5\pi \text{ ft}^2 \approx 39.27 \text{ ft}^2$

   c. What is the measure of the central angle for the sector from part a?

   $45°$

   d. Write the fraction that is equivalent to your answer in part a and that has a denominator of 360. Explain the meaning of both the numerator and denominator using circle terminology.

   Example: $\frac{45}{360}$; 45 is the measure of the central angle of the sector, and 360 is the measure of all the central angles around the circle.

2. Write an equation that will allow Lance to calculate the area of a sector with a central angle of $n°$ and radius r.

   Area of the sector $= \frac{n°}{360°}(\pi r^2)$

CONNECT TO AP

You will work with sectors of circles when you study polar equations in calculus.

My Notes

3. Determine the arc length of each sector of the garden.

arc length $= \frac{\pi d}{8} = \frac{20\pi}{8} \approx 7.85$ ft

4. Write an equation that will allow Lance to calculate the arc length of a sector with a central angle of $n°$ and radius r.

arc length $= \frac{n°}{360°}(2\pi r)$, or $\frac{n°}{180°}(\pi r)$

5. Compare and contrast the arc length of a sector and the area of a sector.

The arc length is the length in linear units of the outer edge of the sector. The area is the number of square units that the sector covers.

6. The architect submits his estimate to Lance. Lance notices that the architect has used the following formulas to calculate the area of each sector and length of each arc in the garden.

$$\frac{\text{degree measure of the central angle}}{360°} = \frac{\text{area of sector}}{\text{area of the circle}}$$

$$\frac{\text{degree measure of the central angle}}{360°} = \frac{\text{arc length of a sector}}{\text{circumference of the circle}}$$

Are the equations you developed in Item 2 and Item 4 equivalent to the architect's equations? Explain your answer.

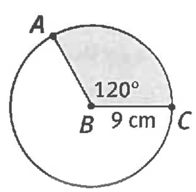
Yes, they are equivalent

Check Your Understanding

7. If a circle has six equal sectors, explain how to determine the measure of the central angle of each sector.
8. Name the term for the sum of all the arc lengths in a circle.
9. **Critique the reasoning of others.** Juliet claims that the length of an arc is the same thing as the measure of an arc. Do you agree? Explain why.
10. The length of an arc in a circle with diameter 10 inches is 1.25π inches. Determine the measure of the arc.
11. The measure of the central angle of a sector is 36° and the radius of the circle is 5 inches. What is the area of the sector?

LESSON 32-2 PRACTICE

12. The length of an arc in a circle with radius 8 inches is 3.2π inches. Determine the measure of the arc.
13. The measure of the central angle of a sector is 60° and the area of the sector is 6π inches^2. Calculate the radius of the circle.
14. **Attend to precision.** Use circle B below to find the following measures.



a. area of circle B
b. area of shaded sector of circle B
c. circumference of circle B
d. length of minor $\overset{\frown}{AC}$

My Notes

7) Divide 360° by 6, which equals 60°

8) circumference

9) No, the length of an arc is measured in linear units, whereas the measure of an arc is measured in degrees.

10) $\frac{x}{360} = \frac{125\pi}{10\pi}$

45°

11) $\frac{36}{360} = \frac{x}{25\pi}$

2.5π inches

12) $\frac{x}{360} = \frac{3.2\pi}{16\pi}$

36°

13) $\frac{60}{360} = \frac{6\pi}{\pi r^2}$

$r = 6$ inches

14a) $81\pi\ cm^2 \approx 254.469\ cm^2$

14b) $27\pi\ cm^2 \approx 84.823\ cm^2$

14c) $18\pi\ cm^2 \approx 56.549\ cm^2$

14d) $6\pi\ cm^2 \approx 18.85\ cm^2$

My Notes

Learning Targets:

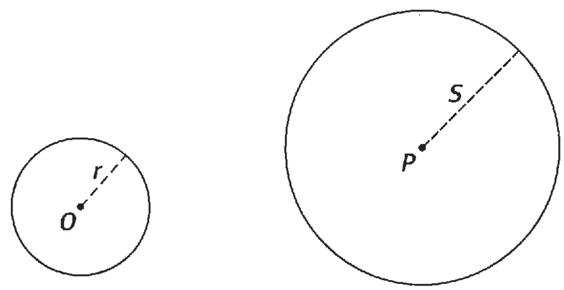
- Prove that all circles are similar.
- Describe and apply radian measure.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Create Representations, Look for a Pattern

Lance likes the design ideas generated by the landscape architect. The architect asks Lance if he will consider making the garden larger in order to include a circular fountain in the center of the garden. Lance is concerned that it will take a long time to update the current sketches and budgets to accommodate the larger garden. The architect assures him that similarity properties apply to all circles and that the changes will be quick and easy.

Lance asks, "Are you sure that all circles are similar?"

The architect draws circle O with radius r to represent the original design of the garden, and circle P with radius s to represent the sketch of the new garden.



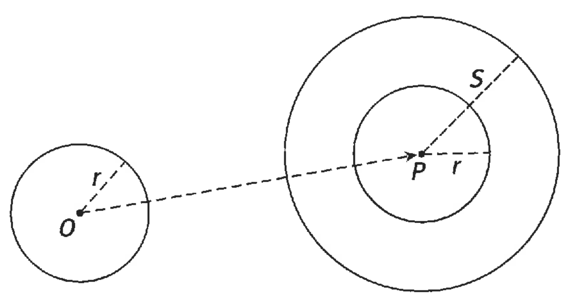
1. **Reason abstractly.** If it is true that any two circles are similar, can one circle be mapped onto the other? Explain your answer.

It is true. Each circle can be mapped onto the other circle by a sequence of transformations.

2. The architect maps the center of circle O onto the center of circle P, as shown in the diagram.

MATH TERMS

Concentric circles share the same center.



a. What transformations are performed?

translation and dilation

b. If O' is the image of circle O, then identify the location of O'.

O′ is located at point P.

My Notes

3. **Make use of structure.** What is the scale factor between the circles?

$\frac{s}{r}$ is the scale factor

4. Two circles share the same center. The radius of one circle is 12, and the radius of the other circle is 8.

a. Describe how the smaller circle was dilated to generate the larger circle.

The smaller circle was dilated using a scale factor of 1.5.

b. Compare the ratios of the circumferences of the two circles.

The circumference of the smaller circle is 16π, whereas the circumference of the larger circle is 24π. The ratio of the circumference of the larger to the smaller circle is 3:2.

c. Compare the ratios of the two areas of the two circles.

The area of the smaller circle is 64π, whereas the area of the larger circle is 144π. The ratio of the area of the larger to the smaller circle is 9:4.

d. Compare the ratios you found in parts b and c to the scale factor. Describe any patterns you notice.

The ratio of the circumference is equal to the scale factor. The ratio of the areas is equal to the square of the scale factor.

Check Your Understanding

5. Circle S is located at (0, 0). It has a radius of 4 units. Circle T is located at (2, 5). It has a radius of 10 units. Describe the sequence of transformations that proves that circle S is similar to circle T.

6. Circle Q, with radius of 18 feet, is reduced by 60 percent. What is the radius of the new circle?

5) translation of $(x+2, y+5)$ and a dilation of 2.5.

6) $\frac{100-60}{100} = \frac{x}{18}$

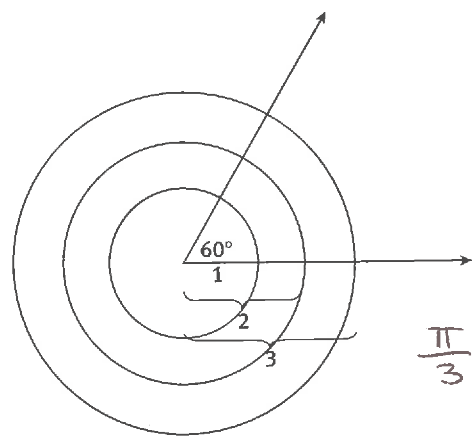
$x = 7.2$ feet

My Notes

To finalize the design of the garden, the landscape architect is using a landscaping software program. To operate it correctly, the architect has to enter some of the angles in radians and some of the angles in degrees.

Recall the formula for the arc length s of an arc with measure $m°$ and radius r: $s = \frac{m°}{360°}(2\pi r)$. This formula can be applied to concentric circles.

7. Given the three concentric circles below, with radius 1 unit, 2 units, and 3 units, do the following.
   a. Calculate the arc lengths of each of the three arcs formed by the 60° angle.



$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}$

   b. Complete the following table for the three circles. Do not use an approximation for π.

Circle	Measure of Central Angle	Radius	Arc Length	Arc Length: Radius
1	60°	1	$\frac{\pi}{3}$	$\frac{\pi}{3}$
2	60°	2	$\frac{2\pi}{3}$	$\frac{\pi}{3}$
3	60°	3	$\frac{3\pi}{3}$	$\frac{\pi}{3}$

8. **Express regularity in repeated reasoning.** What pattern do you notice in the table?

For a central angle of 60°, the ratio of arc length to radius is always $\frac{\pi}{3}$.

The architect points out that for a given fixed angle measure m, the length of the arc cut off by the central angle is proportional to the length of the radius. This constant of proportionality, $\frac{m^\circ}{360^\circ}(2\pi) = \frac{m^\circ}{180^\circ}\pi$, defines the ***radian measure*** of the angle.

My Notes

radians = degrees $\cdot \frac{\pi}{180}$

degrees = radians $\cdot \frac{180}{\pi}$

CONNECT TO SCIENCE

Units are an important part of scientific values and calculations. Scientists use one system of measurement—the SI system—to make it easier to communicate among themselves. The radian is the SI unit for angular measurement. Scientists often need to convert units using conversion ratios, as you may have done in your science classes.

Example A

Converting degrees to radians.

Convert 150° to radians.

Substitute $m = 150$ into the expression $\frac{m^\circ}{180^\circ}\pi$.

$$\frac{m^\circ}{180^\circ}\pi = \frac{150^\circ}{180^\circ}\pi = \frac{5\pi}{6}$$

So, $150^\circ = \frac{5\pi}{6}$ radians.

Example B

Converting radians to degrees.

Convert $\frac{\pi}{5}$ radians to degrees.

Multiply radians by $\frac{180^\circ}{\pi}$.

$$\frac{180^\circ}{\pi}\left(\frac{\pi}{5}\right) = 36^\circ$$

So, $\frac{\pi}{5}$ radians $= 36^\circ$.

Try These A-B

a. Complete: $20^\circ =$ $\pi/9$ radians

b. Complete: $270^\circ =$ $3\pi/2$ radians

c. Complete: $\frac{2\pi}{3}$ radians $=$ 120 °

d. Complete: $\frac{\pi}{9}$ radians $=$ 20 °

a) $20^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{9}$

b) $270^\circ \cdot \frac{\pi}{180} = \frac{3\pi}{2}$

c) $\frac{2\cancel{\pi}}{3} \cdot \frac{180}{\cancel{\pi}} = 120$

d) $\frac{\cancel{\pi}}{9} \cdot \frac{180}{\cancel{\pi}} = 20$

My Notes

9) Alex is correct. Circles are only congruent if they have the same diameter.

10) 0; π

11a) No; the circles have different radii.

11b) No; the circles have different centers, which means the circles have been translated, and circle P is smaller than circle O, so that means it has been dilated

11c) Circle O can be mapped to circle P using first a translation and then a dilation.

12a) 90°
12b) 45°

13a) $\frac{5\pi}{12}$

13b) $\frac{5\pi}{6}$

13c) $\frac{5\pi}{3}$

Check Your Understanding

9. Alex states that all congruent circles are similar. Ethan states that all similar circles are congruent. With whom do you agree? Justify your reasoning.

10. What radian measure is equivalent to 0°? 180°?

LESSON 32-3 PRACTICE

11. **Make sense of problems.** Circle *O* is located at (2, 1). It has a radius of 4 units. Circle *P* is located at (−1, −1). It has a radius of 2 units.
- **a.** Are the circles congruent? How do you know?
- **b.** Can you prove that circle *O* and circle *P* are similar using only one similarity transformation? Explain.
- **c.** How can you prove that circle *O* and circle *P* are similar using more than one similarity transformation?

12. What degree measure is equivalent to the following radian measures?
- **a.** $\frac{\pi}{2}$
- **b.** $\frac{\pi}{4}$

13. What radian measure is equivalent to the following angle measures?
- **a.** 75°
- **b.** 150°
- **c.** 300°