Pyramids and Cones

Perfect Packaging

Lesson 35-1 Surface Area of Pyramids and Cones

My Notes

Learning Targets:

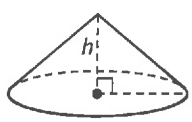
- Solve problems by finding the lateral area or total surface area of a pyramid.
- Solve problems by finding the lateral area or total surface area of a cone.

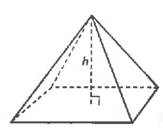
SUGGESTED LEARNING STRATEGIES: Create Representations, Visualization, Think-Pair-Share

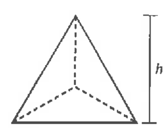
The way a product is packaged is an important marketing element for manufacturers, and in today's world, it is an important environmental concern for consumers.

Luis is leading a committee to change some of the current packaging materials used by his company that have been deemed nonrecyclable to more environmentally friendly materials. Many of the environmentally friendly materials, however, do not have the same strength or durability as the packaging materials currently in use. The committee realizes that if the materials are changed, then the shape of the actual package will likely have to change to ensure strength and durability.

The committee consists of representatives from the marketing department and the advertising department, a product engineer, and a packaging engineer. To begin, the marketing representative provides the committee with feedback that consumers are attracted to products packaged in pyramid- and cone-shaped containers.

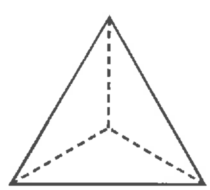






Surface area is a critical concern for the advertising representative, as it dictates how much space is available for print. Before the committee goes any further, the advertiser wants to determine the surface area of a cone and pyramid.

1. Consider the *tetrahedron*, a special type of pyramid with an equilateral triangle for a base and equilateral triangles for each of its three sides. Suppose the length of each edge of the container is 8 ft.



a. Find the area of each face.

$16\sqrt{3}$ ft^2, or 27.713 ft^2

b. Find the total surface area of the four faces.

(multiply by four)

$64\sqrt{3}$ ft^2, or 110.851 ft^2

MATH TIP

The formula for finding the area of an equilateral triangle, given the length of a side s, is $A = \frac{s^2\sqrt{3}}{4}$.

My Notes

c. Use the formula for calculating the area of an equilateral triangle to derive a formula for the *total surface area* of one of these pyramids in terms of the length of an edge, *s*.

total surface area of pyramid = 4(area of one face)

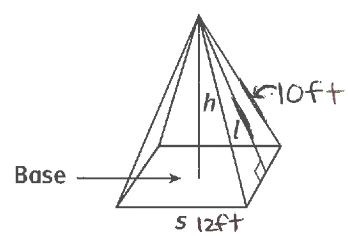
$4\left(\frac{s^2\sqrt{3}}{4}\right) = s^2\sqrt{3}$ units^2

d. The packaging engineer provides a 27.75 sq. yd sample of environmentally friendly material to create a model of this tetrahedron-shaped container. Find the dimensions of the pyramid.

4ft $\left(27.75 = s^2\sqrt{3}\right)$

$s \approx 4$ft

2. Consider the square pyramid. Suppose each side of the square base of the pyramid is 12 ft, and the length of each lateral edge is 10 ft.



a. Determine the perimeter and the area of the base.

P=48ft; A = 144ft^2

b. How can you determine the height of a triangular face?

$h^2 = 10^2 - 6^2$; Pythagorean Theorem
(6 is half of 12.)

c. The height of a triangular face is the ***slant height***. Draw one of the slant heights, and determine its length. Then calculate the area of one of the triangular faces.

slant height $= 10^2 - 6^2 = \sqrt{64} = 8$ft

area of triangular face $= \frac{1}{2}(12)(8) =$ 48ft^2

d. The **lateral area of a pyramid** is the sum of the areas of the lateral faces. Calculate the lateral area of the pyramid.

LA = (48)(4) = 192ft^2

4 faces

MATH TERMS

The **slant height** of a regular pyramid, such as a square pyramid, is the distance between the vertex and the base edge of the pyramid. It is the altitude of a lateral face.

My Notes

e. If ℓ represents the slant height of the pyramid, create a formula for finding the lateral area, *LA*, of a right pyramid in terms of ℓ and the perimeter of the base, *p*.

$LA = \frac{1}{2}p\ell$

f. The **total surface area of a pyramid** is the sum of the lateral area and the area of the base. Calculate the total surface area for the square pyramid.

$SA = LA + B$
$= 192 + 144$
$= 336 ft^2$

g. Write a formula for calculating the total surface area, *SA*, of a right pyramid in terms of the lateral area, *LA*, and the base area, *B*.

$SA = LA + B$

3. Use your formulas in Items 2e and 2g to find the lateral area and total surface area for a regular tetrahedron, each of whose edges is 8 feet in length.

$LA = 48\sqrt{3}\ ft^2$ (or $83.138 ft^2$)
$SA = 64\sqrt{3} ft^2$ (or $110.851 ft^2$)

Check Your Understanding

4. Name the shapes of the lateral faces of a regular pyramid.

5. Can a pyramid ever have quadrilaterals for lateral faces? Explain.

6. Each side of the base of a square pyramid is 5 ft. The slant height of the pyramid is 4 ft.
a. Calculate the lateral area of the pyramid.
b. What is the surface area of the pyramid?

7. Each edge of a regular tetrahedron is 12 in.
a. Calculate its lateral area.
b. What is its surface area?

4) isosceles triangles

5) No; pyramids only have one vertex so all faces are triangular.

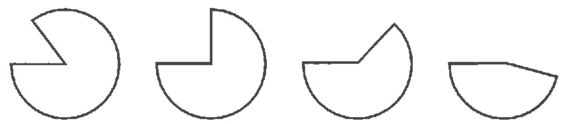
6a) $LA = 4(\frac{1}{2}(5)(4)) = 40 ft^2$
$SA = LA + B$
$= 40 + (5)(5)$
$= 65 ft^2$

7a) area of each face $= \frac{s^2\sqrt{3}}{4}$
$LA = 3\left(\frac{12^2\sqrt{3}}{4}\right) = 108\sqrt{3}\ in^2$

$SA = 144\sqrt{3}\ in^2$

My Notes

The advertising representative would like to have a formula for calculating the surface area for a cone-shaped package, in terms of the dimensions. She knows the formula for the area of the base of a cone, which is a circle. It's the lateral area that is troubling her. The packaging engineer guides her through an explanation by displaying several patterns for the curved surface of a cone, the **lateral surface**.



8. **Use appropriate tools strategically.** The engineer points out that each pattern is actually a sector of a circle. Find the patterns for the curved surface of several cones on the worksheet your teacher provided. Cut out each pattern and build the lateral surface for each cone. Trace each circular base on a sheet of paper and answer the questions below.

   a. As the lateral area of the cone decreases, what happens to the height of the cone?

   The height increases

   b. As the lateral area of the cone decreases, what happens to the radius of the base?

   The radius decreases.

   c. As the lateral area of the cone decreases, what happens to the slant height of the cone?

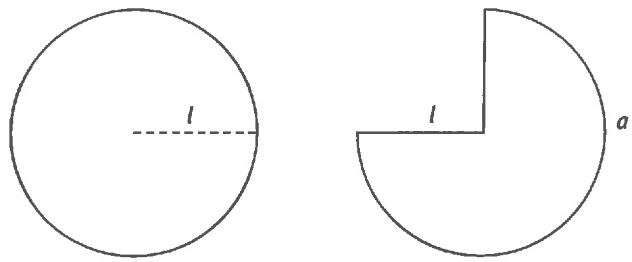
   The slant height remains constant.

   d. Which part of the cone corresponds to the radius of the sector?

   the slant height

My Notes

The engineer starts each pattern for the lateral surface with a large circle with radius l. Then he cuts out a sector, as shown below.



9. **Make sense of problems.** Use the figures above to respond to each of the following.
   a. Write an expression for the area and circumference of the circle in terms of l and π.
   $A = l^2\pi$; $C = 2\pi l$
   b. Write the ratio of the area of the sector A_S to the area of the circle.
   $\frac{A_s}{l^2\pi}$
   c. Write the ratio of the length of the arc of the sector a to the circumference of the circle.
   $\frac{a}{2\pi l}$
   d. Explain why the two ratios in parts b and c can be used to create a proportion.
   They both represent the same fraction of the circle $\frac{n}{360}$.
   e. Write the proportion from part d, and solve this proportion for A_S. Simplify your answer.
   $A_s = \frac{al}{2}$
   f. How does a, the length of the arc of the sector, relate to the base of the cone?
   a is the circumference of the base of the cone; $a = 2\pi r$
   g. Write an expression for the lateral area of a cone, LA, in terms of l, the radius of the base, r, and π.
   $LA = \pi r l$
   h. Write a formula for calculating the total surface area, SA, in terms of the lateral area, LA, and the base area, B.
   $SA = LA + B$

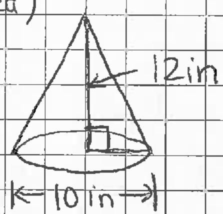
My Notes

10) The base of a cone is a circle, whereas the base of a pyramid is a polygon

11a) 35π cm^2

11b) 60π cm^2

12a)



12b) LA = 65π ft^2
12c) SA = 90π ft^2

13) Use the area of the base to find the radius. Use the lateral area to find the slant height. Finally, substitute the slant height and the radius into the Pythagorean Theorem to determine the height: 15ft

14) LA = 240 ft^2; SA = 384 ft^2

15) 396 mm^2

16) 153π ft^2

17) LA = $64\sqrt{2}\pi$ mm^2
SA = $(64\sqrt{2} + 64)\pi$ mm^2

18) 444π ft^2

Check Your Understanding

10. Compare the formulas for surface area of cones and pyramids. Explain why the base area of a cone is not computed using the same formula as the base area for a pyramid.

11. The diameter of the base of a cone is 10 centimeters. The slant height of the cone is 7 centimeters. r = 5 cm
    a. Determine the lateral area of the cone.
    b. What is the surface area of the cone?

12. A plastic cone-shaped package is 10 inches in diameter and 12 inches high.
    a. Make a sketch of the pyramid and label the dimensions.
    b. Find the lateral area.
    c. Compute the total surface area of the package.

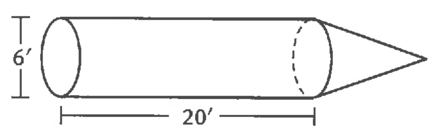
13. Describe a step-by-step method you can use to find the height of a cone if you know the lateral area and the base area. Then use your method to find the height of a right cone whose lateral area is 136π ft^2 and base area is 64π ft^2.

LESSON 35-1 PRACTICE

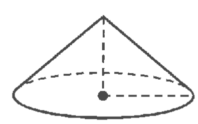
14. Calculate the lateral area and surface area of a pyramid whose height is 8 ft and whose base is a square with 12 ft sides.

15. Find the lateral area of a right pyramid whose slant height is 18 mm and whose base is a square with area 121 mm^2.

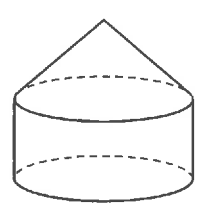
16. **Reason quantitatively.** Calculate the surface area of the cylindrical rocket and "nose cone" if the slant height of the nose cone is 8 ft.



17. Find the lateral area and surface area for a right cone whose slant height forms a 45° angle with the base and whose radius is 8 mm.



18. The product engineer shows the committee a product he is working on that is cylindrical in shape with a conical top. Find the total surface area of the product given the diameter is 24 ft, the height of the cylindrical portion is 6 ft, and the height of the conical portion is 5 ft.



My Notes

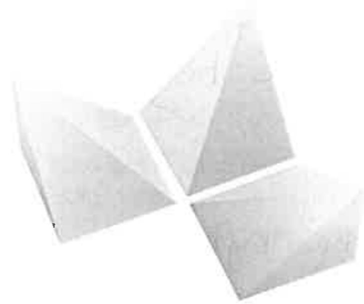
Learning Targets:

- Solve problems by finding the volume of a pyramid.
- Solve problems by finding the volume of a cone.

SUGGESTED LEARNING STRATEGIES: Visualization, Group Presentation, Self Revision/Peer Revision, Think-Pair-Share, Interactive Word Wall

For any new package design, the packaging engineer needs to confirm it can meet the volume requirements for the product. Luis asks the packaging engineer to explain how volume is computed for packages shaped like a pyramid or cone.

Consider the three potential package designs below. Each pyramid has the same base area and the same height.



1. **Model with mathematics.** The packaging engineer manipulates the pyramid into the figure shown. What figure do the three pyramids form?
a cube

My Notes

MATH TIP

There is a postulate that states that pyramids with the same base area and the same height have the same volume.

2. A cube is a type of prism.

a. What is the formula for volume of a prism?

$V = Bh$

b. How many pyramids form the cube?

3

c. Do the pyramids within the cube have the same volume? How do you know?

Yes, they have the same base area and the same height.

d. Describe how the formula from Item 2a can be adapted to create a formula for the volume of one of the square-based pyramids.

Divide the right side of the formula by 3.

e. Write the formula for the volume of one of the square-based pyramids.

$V = \frac{Bh}{3}$ or $\frac{1}{3}Bh$

MATH TERMS

Cavalieri's Principle states that two objects with the same height and the same cross-sectional areas at every height have the same volume.

The *height of an oblique pyramid* is the length of the perpendicular segment from its vertex to the plane of the base.

3. Describe how ***Cavalieri's Principle*** can be used to justify the formula in Item 2e for any square-based pyramid. Remember to use complete sentences and words such as *and, or, since, for example, therefore, because of, by the,* to make connections between your thoughts.

If two pyramids have the same height and the same cross-sectional areas, then the volumes are equal. Since the formula is true for these particular square-based pyramrds, then it is also true for any square pyramid of the same height and cross-sectional areas.

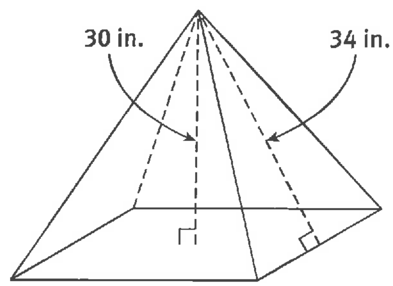
4. Critique the reasoning of others. Keith states that the formula in Item 2e does not apply to a pyramid with *any* polygonal base. Angela states that the formula does apply, and that only the formula used to compute *B* will be different. With whom do you agree and why?

Angela. Cavalieri's Principle states that the formula used for B does change because the base area is computed differently based on its shape.

My Notes

Example A

Find the volume of a package shaped like a square pyramid, with the given measures.



Step 1: Find the length of the base.

The distance between the height of the pyramid and the slant height is one-half the base. Use the Pythagorean Theorem to find this distance.

$$d^2 = 34^2 - 30^2$$
$$d = 16$$

The length of the base is 2(16) = 32 in.

Step 2: Use the volume formula for a pyramid, $V = \frac{1}{3}Bh$, and substitute known values.

$$V = \frac{1}{3}Bh$$
$$V = \frac{1}{3}(32)(32)(30)$$
$$V = 10{,}240 \text{ in.}^3$$

Solution: The volume of the pyramid is 10, 240 in.^{3}

Try These A

a. The height of a pyramid is 24 ft. The area of the base is 22 ft^2. Compute the volume of the pyramid.

$V = \frac{1}{3}Bh = \frac{1}{3}(22)(24) = 176 ft^3$

b. The volume of a pyramid is 220 cm^3 and the base area is 10 cm^2. What is the height of the pyramid?

$220 = \frac{1}{3}(10)(h)$

$h = 66 cm$

c. The slant height of a square pyramid is 10 feet. One side of the base of the pyramid is 12 feet. What is the volume of the pyramid?

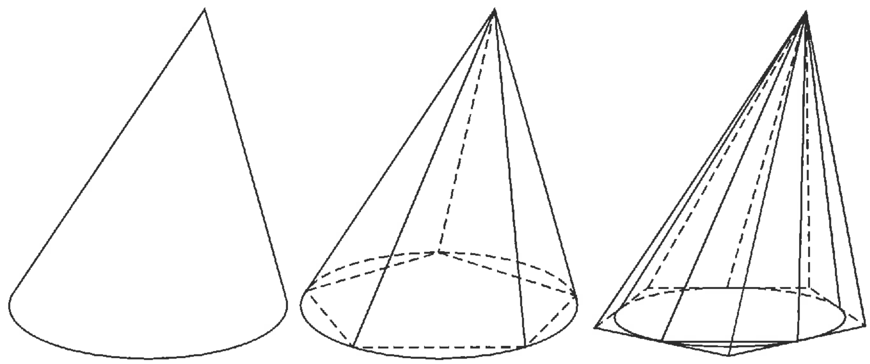
$V = 384 ft^3$

$(B = 144 ft^2;\ h = 8 ft)$

use Pythagorean Theorem

My Notes

Now, consider a package design in the shape of a cone. The packaging engineer develops the volume formula by inscribing polygonal pyramids of varying bases in a cone.



5. Write the formula for the volume of a pyramid.

$V = \frac{Bh}{3}$ or $\frac{1}{3}Bh$

6. Examine the base of each cone. As the number of sides increases in the polygonal pyramid, is the space between the pyramid base and the circular base of the cone increasing or decreasing?

decreasing

7. Reason abstractly. Suppose the number of sides of the polygonal pyramid inscribed in the cone continued to increase. What shape would the base of the pyramid resemble?

a circle

8. Write the area formula for a circle.

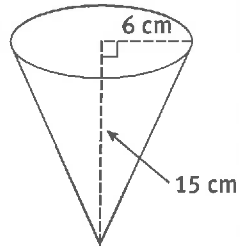
$A = \pi r^2$

9. Use the formulas in Items 5 and 8 to write the formula for volume of a cone.

$V = \frac{1}{3}\pi r^2 h$ (or $\frac{\pi r^2 h}{3}$)

10. Compute the volume of the cone with a radius of 6 cm and a height of 15 cm, to the nearest tenth.

$V = \frac{1}{3}\pi(6)^2(15) = 565.5 cm^3$



Check Your Understanding

11. A container of nuts in the shape of a right cone has a diameter of 18 cm. The height of the container is 7 cm. What is the volume of nuts that fits in the container?

12. Compare and contrast the volume formulas for cones and pyramids.

13. In terms of units of measure, how does the surface area of a cone differ from the volume of a cone?

14. A perfume bottle in the shape of a right cone has a diameter of 6 in. and a slant height of 5 in. How much perfume can the bottle hold, to the nearest cubic inch?

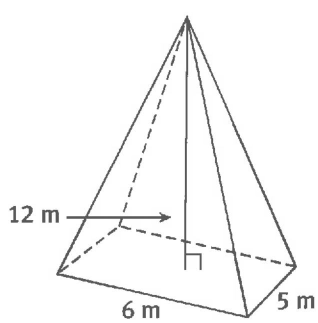
LESSON 35-2 PRACTICE

15. The height of a cone is 10 ft. The area of the base is 22 ft^2. Compute the volume of the cone.

16. The volume of a cone is 80 cm^3 and the base area is 5 cm^2. What is the height of the cone?

17. The volume of a right cone is 400 ft^3 and the height is 25 ft. Find the slant height of the cone, to the nearest tenth.

18. The base of a right pyramid has dimensions of 5 m by 6 m. The height of the pyramid is 12 m. What is the volume of the pyramid?



19. **Make sense of problems.** Peanuts are sold at the baseball game for \$3.75. The peanuts are packaged in boxes shaped as rectangular prisms. The height of the box is 7 inches and the base of the box has an area of 3 in.^{2}. For the tournament games, management decided to package the peanuts in a pyramid design rather than as a prism. The pyramid has the same height and base area as the prism.

a. Compare the volumes of the prism and the pyramid.

b. Determine a price for the peanuts in the pyramid design that is proportionally fair to the original price. Explain your answer.

My Notes

11) $593.46 cm^3$
($r = 9cm$)

12) In both formulas, you are finding one-third of the base area times the height. In the cone formula, the base is always a circle, so the area is computed using πr^2. In a pyramid, the base is always a polygon, so the area is computed using the appropriate formul

13) The surface area of the cone describes the number of square units to cover the entire surface of the cone. The volume describes the number of cubic units that can fit inside the cone.

14) $r = 3, h = 4$
$V = 38 in^3$

15) $V = \frac{1}{3}Bh$ $B = 22, h = 10$
$V = 73\frac{1}{3} ft^2$

16) $V = \frac{1}{3}Bh$
$80 = \frac{1}{3}(5)(h)$
$h = 48cm$

17) 25.3ft

18) $120 m^3$

19a) $21ft^3$ for prism; $7ft^3$ for pyram
The volume of the prism is three times greater.

19b) \$1.25, since the volume of the pyramid is one-third of the volume of the prism with the same base and height.

My Notes

Learning Targets:

- Apply concepts of density in modeling situations.
- Apply surface area and volume to solve design problems.

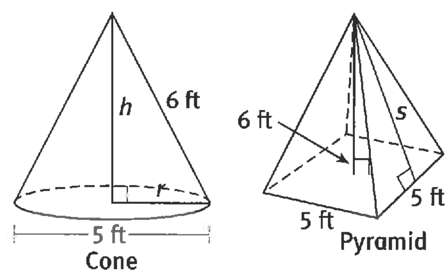
SUGGESTED LEARNING STRATEGIES: Group Presentation, Self Revision/Peer Revision, Think-Pair-Share, Interactive Word Wall

Luis has learned that before they make any decisions about changing packaging materials, he needs to make sure he is within budget for the cost of the packaging materials and the cost of shipping the product.

The new material under consideration for packaging food products costs \$0.45 per square foot. The shipping cost is \$1.90 per pound. The table below shows the density for four different food products packaged in bulk and shipped from Luis's company.

Material	Density (lb/ft^3)
Shelled almonds	30
Apple seeds	32
Coffee beans	22
Soybean hulls	6

Luis is analyzing the cost of the following two package designs to determine which package makes the most sense to use for coffee beans.



1. **Model with mathematics.** Consider the cone. The diameter is 5 ft and the slant height is 6 ft.
   a. Determine the surface area of the cone. Then find the cost of the packaging material.
   SA = 66.7588 ft²; \$30.04
   b. Determine the volume of the cone.
   V = 35.6987 ft³
   c. What is the relationship among volume, density, and mass?
   $\text{density} = \frac{\text{mass}}{\text{volume}}$
   d. Find the mass of the cone filled with coffee beans.
   785.3714 lb.
   e. Find the shipping cost of the cone.
   \$1492.21
   f. What is the total cost for packaging and shipping the cone-shaped package filled with coffee beans?
   \$1522.25

My Notes

2. Make a prediction as to whether the total cost for packaging and shipping the pyramid-shaped package will be greater than, less than, or the same as the cone-shaped package. Explain your reasoning.

Answers will vary.

3. Determine the validity of your prediction by analyzing the pyramid package.

a. Find the cost of the packaging material for the pyramid design.

$90ft^2$ ($0.45 per ft^2) = $40.50

b. Find the shipping cost of the pyramid.

$50ft^3$ (22 lb/ft^3)($1.90) = $2090

c. What is the total cost for packaging and shipping the cone-shaped package?

$2130.50

4. Compare the ratio of total cost to ship each package to volume shipped. How did your prediction in Item 2 compare to your findings?

$$\text{cone} = \frac{\$1522.25}{35.6987 ft^3} = \$42.64/ft^3$$

$$\text{pyramid} = \frac{\$2130.5}{50 ft^3} = \$42.61/ft^3$$

5. Which food product in the table would be the least expensive to ship? Explain how you determined your answer.

The soybean hulls; they have the least density, so the overall mass for a given volume would be less.

My Notes

6) No, the density of apple seeds is greater than that of coffee beans, so the apple seed package has the greater mass.

7) 3.5 lb.

8) 1 pound

9a) the cylinder will have greater mass because it has greater volume.

9b) The volume of the cone is about $\frac{1}{3}$ of the volume of the cylinder.

9c) Cone: 301.59 lb.
Cylinder: 904.32 lb.

Yes, the mass of the cylinder is about three times greater than the mass of the cone. They were filled with the same substance, so the measures are reasonable.

10) No; it is likely due to the difference in densities that the Styrofoam will not have the strength to handle the mass of the sandstone.

Check Your Understanding

6. Apple seeds are packaged in a container shaped like a square pyramid. Coffee beans are packaged in an identical container. Do the filled packages have the same mass? If not, explain how you can determine the product with the greater mass.
7. Shelled almonds are packaged in a cylindrical container with a 0.5-foot diameter and a height of 0.6 ft. What is the approximate mass of the almonds that fill the container, to the nearest tenth of a pound?

LESSON 35-3 PRACTICE

8. Butter has a density of 54 lb/ft^3. It is packaged into a 5 in.-by-1.27 in.-by-1.27 in. cardboard prism. One package of butter contains four identical prisms. What is the approximate mass of one package of butter?
9. Two containers, a cone and a cylinder, both have a height of 6 ft. The bases have the same diameter of 4 ft. Both containers are filled with shredded paper that has a density of 12 lb/ft^3.
   a. How do you know which container has the greatest mass?
   b. Make a generalized statement about the difference in volume of a cone and a cylinder given the same diameter and height.
   c. Determine the mass of each container. Are your answers reasonable? Explain how you know.
10. **Construct viable arguments.** A company needs to package crushed sandstone to send to a customer. Sandstone has a density of 1281 kg/m^3. The customer asked if the company could send the sandstone in a Styrofoam container. Styrofoam has a density of 70 kg/m^3. Should the company use a Styrofoam container to send the sandstone? Explain.