Deriving Area Formulas

Shape Up

Lesson 30-1 Areas of Rectangles and Parallelograms

My Notes

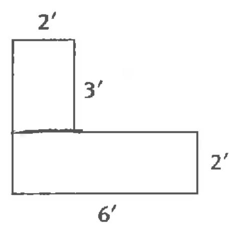
Learning Targets:

- Solve problems using the areas of rectangles, parallelograms, and composite figures.
- Use coordinates to compute perimeters and areas of figures.

SUGGESTED LEARNING STRATEGIES: Use Manipulatives, Quickwrite, RAFT

Lisa works in the billing department of A Cut Above, a company that builds custom countertops and tabletops. A new customer has contracted A Cut Above to build the tabletops for a theme restaurant, Shape Up.

Each tabletop costs $8.50 per square foot. Lisa's job is to calculate the area, compute the charge, and bill each customer properly. The tabletops, made with laminated wood, are delivered to A Cut Above in different rectangular sizes. Before there are any cuts, a cardboard template is made to use as a guide. Lisa uses the templates to investigate the areas. Shown is one of the templates for which Lisa must find the area.



1. a. Find the area of this shape. Include units of measure in your answer. Describe the method used.

   $18 ft^2$; divide the shape into two rectangles

   b. How much should Lisa charge the customer for this tabletop?

   \$153; $(18)(8.50) = 153$

In keeping with the theme of the restaurant, the customer wants to include tables in the shapes of triangles, parallelograms, and trapezoids. Lisa decides to investigate area formulas by first finding the formula for the area of a parallelogram.

DISCUSSION GROUP TIPS

As you work in groups, read the problem scenario carefully and explore together the information provided. Discuss your understanding of the problem and ask peers or your teacher to clarify any areas that are not clear.

2. **Use appropriate tools strategically.** Use the rectangle provided by your teacher.
   - Pick a point, not a vertex, on one side of the rectangle. Draw a segment from the point to a vertex on the opposite side of the rectangle.
   - Cut along the segment.
   - Put the two figures together to form a parallelogram.

   a. Explain why the area of a parallelogram is the same as the area of rectangle. The same amount of paper was used to create both the rectangle and parallelogram.

   b. **Critique the reasoning of others.** Nelson and Rashid, two tabletop designers, are discussing how to calculate the area of any parallelogram. Nelson suggests that they multiply the lengths of two consecutive sides to find the area, as they do for rectangles. Rashid claims this will not work. Explain which designer is correct and why.

   Rashid is correct. To find the area of a parallelogram, you need to use the perpendicular height, which is not a side length unless the parallelogram is a rectangle.

My Notes

Area of a parallelogram

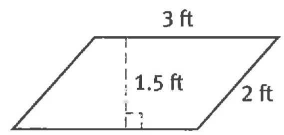
$A = bh$

NOTE: The base and height in any geometric formula must be perpendicular !!

3. A scale drawing of the parallelogram tabletop template is shown below. Determine the charge for making this tabletop. Show the calculations that led to your answer.

$38.25

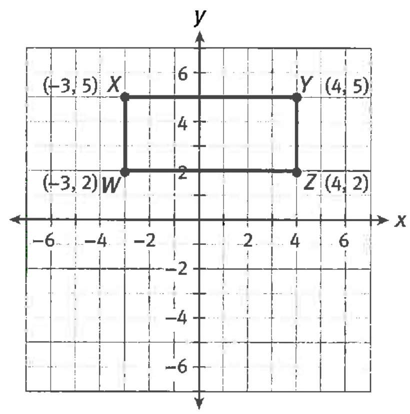
$(4.5)(8.50) = 38.25$



$\text{Area} = (3)(1.5)$
$= 4.5\ ft^2$

Lisa finds the template method somewhat time-consuming and asks Rashid if there is a different way to determine the measures of the tabletops. Rashid tells Lisa that she can use a coordinate plane to determine the perimeter and area of a tabletop.

4. A diagram showing a rectangular tabletop is plotted on the coordinate plane.



Explain how to use the coordinates to find the length and width of the rectangular tabletop. Then use the dimensions to compute the perimeter and area of the tabletop. Show the formulas that you used.

To find XY, subtract the x-coordinates of point X and point Y: $4-(-3) = 7$
To find YZ, subtract the y-coordinates of point Y and point Z: $5-2 = 3$

$\text{Perimeter} = 2(l+w)$
$= 2(7+3)$
$= 20$ units

$\text{Area} = lw = (7)(3)$
$= 21\ \text{units}^2$

5. Points $F(-3, 4)$, $G(2, 2)$, $H(2, -4)$, and $J(-3, -2)$ are the vertices of parallelogram $FGHJ$.

a. Draw the parallelogram on the grid.

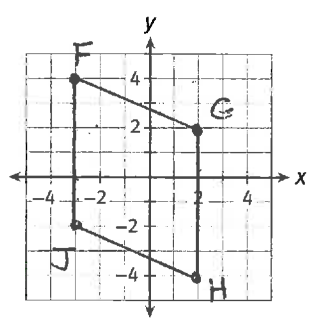
b. Determine the perimeter of the parallelogram to the nearest tenth.

22.8 units

c. What is the area of the parallelogram?

$b = 6,\ h = 5$

30 square units

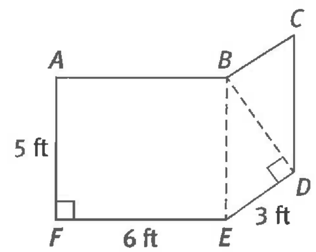


$FG = \sqrt{5^2+2^2} = \sqrt{29}$
$JH = FG = \sqrt{29}$
$FJ = 6$
$GH = 6$

My Notes

Lisa realizes that she can apply what she has learned to determine the measures of tabletops that are formed using a combination of shapes.

6. A diagram of a custom tabletop is shown. The tabletop is a ***composite figure*** that consists of a rectangle and parallelogram. Lisa needs to calculate the cost of the tabletop.



MATH TERMS

A **composite figure** is made up of two or more simpler shapes, such as triangles, rectangles, or parallelograms. The word *composite* means made up of distinct parts. For example, in geology, a composite volcano is made up of alternating layers of lava and rocks.

The key to solving a composite figure is to break it into figures of shapes that we have formulas for.

a. What additional dimension(s) does Lisa need to know in order to find the area of the tabletop? How can she find this? Be sure to use properties or theorems to justify your answer.

She needs to know the height, BD, of the parallelogram EBCD. BE = 5ft, therefore by the Pythagorean Theorem, BD = 4 feet.

b. What is the cost of the tabletop? Show your work.

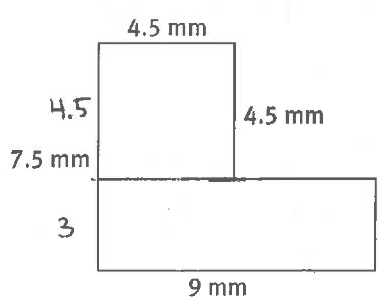
Area of rectangle $= lw = 5(6) = 30\text{ ft}^2$
Area of parallelogram $= bh = 3(4) = 12\text{ ft}^2$
Total area $= 42\text{ ft}^2$
Cost of tabletop $= 42(8.50) = \$357$

Check Your Understanding

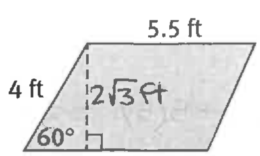
7. Find the area of the composite figure shown. 47.25 mm^2

8. Determine the area and perimeter of a rectangle with length 12 cm and diagonal length 13 cm.

9. Determine the area and perimeter of the parallelograms shown. Round the answers to the nearest tenth.

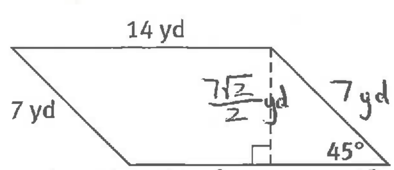


a.



$P = 19\text{ ft}$; $A = 19.1\text{ ft}^2$

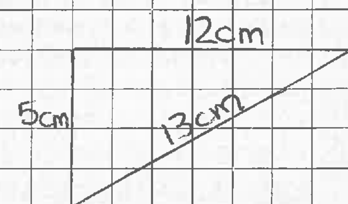
b.



$P = 42\text{yd}$; $A = 69.3\text{yd}^2$

7) Area of square $= (4.5)(4.5) = 20.25\text{ mm}^2$
Area of rectangle $= 27\text{ mm}^2$
Total area $= 47.25\text{mm}^2$

8)



$P = 34\text{cm}$; $A = 60\text{cm}^2$

My Notes

10) $P = 30ft$
$A = 32ft^2$

11) $P = 16ft$
$A = 11.2ft^2$

12) $P = 16ft$
$A = 15ft^2$

13) Table 1: 32(8.50)
\$272

Table 2: 11.2(8.50)
\$ 95.20

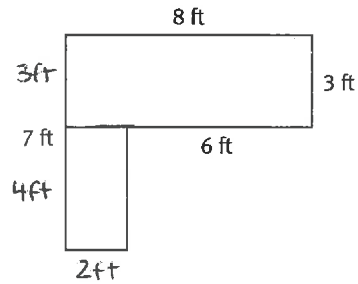
Table 3: 15(8.50)
\$ 127.50

14) The height is 5 units and the base is 9 units

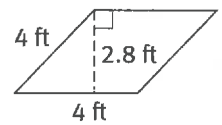
LESSON 30-1 PRACTICE

Lisa was given the list of all of the shapes and necessary dimensions of some tabletops. In Items 10–12, determine the perimeter and area of each tabletop.

10. Table 1



11. Table 2

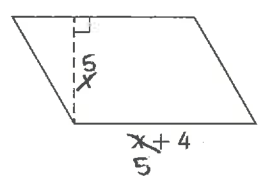


12. Table 3: Figure *QRST* with vertices $Q(-1, 4)$, $R(-1, 7)$, $S(4, 7)$, and $T(4, 4)$. Each unit of the coordinate plane represents one foot.

13. Reason quantitatively. Given that the charge for each tabletop is \$8.50 per square foot, create an itemized invoice to show the cost of each tabletop from Items 10–12 and the total cost.

14. The area of the parallelogram shown is 45 square units. Find the height and base of the parallelogram.

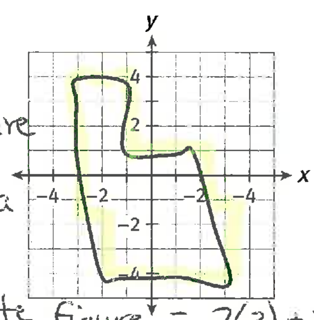
$A = bh$
$45 = (x+4)(x)$
$x^2 + 4x = 45$
$x^2 + 4x - 45 = 0$



$(x+9)(x-5) = 0$
$x + 9 = 0$
$\cancel{x = -9}$
$x - 5 = 0$
$x = 5$

15. Explain how to use a composite figure to estimate the area of the irregular figure shown on the coordinate plane.

The figure is close to a composite figure composed of a rectangle and a parallelogram.



Area of composite figure = 2(3) + 5(5) = 31 square units
Area is approximately 31 square units

DISCUSSION GROUP TIPS

If you do not understand something in group discussions, ask for help or raise your hand for help. Describe your questions as clearly as possible, using synonyms or other words when you do not know the precise words to use.

Learning Targets:

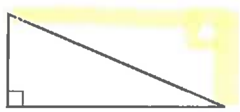
- Solve problems using the areas of triangles and composite figures.
- Use coordinates to compute perimeters and areas of figures.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Group Presentation, Identify a Subtask, Use Manipulatives, Quickwrite

Lisa asks Rashid to help her understand how to determine the area of triangular-shaped tabletops. Lisa and Rashid start the investigation with right and isosceles triangles.

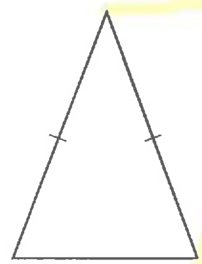
1. Use the given triangles below to create quadrilaterals. Using what you know about these quadrilaterals, explain why the formula for the area of a triangle is Area $= \frac{1}{2} \cdot$ base $\cdot$ height.

a.



The triangle is half of a rectangle

b.



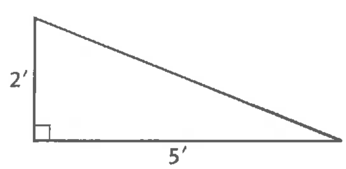
The isosceles triangle is half of a parallelogram.

2. **Reason abstractly.** Compare and contrast the steps required to find the area of a right triangle and an isosceles triangle whose side lengths are given.

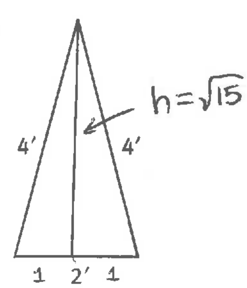
They both use the same formula; however, the height of the isosceles triangle must be drawn in and its length determined using the Pythagorean Theorem.

3. Find the area and subsequent charge for each tabletop below.

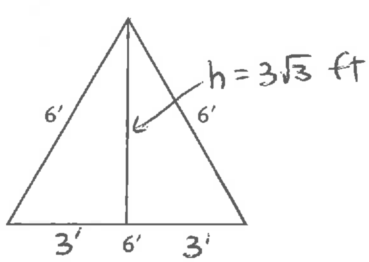
a.



b.



c.



My Notes

Area of triangle:

$A = \frac{bh}{2}$ or $\frac{1}{2}bh$

NOTE: If you know the length of each side of the triangle, try Heron's formula

$A = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{a+b+c}{2}$

MATH TIP

An isosceles triangle has two congruent sides and two congruent angles.

3a) $A = \frac{1}{2}(5)(2) = 5 \text{ ft}^2$
$5(8.50) = \$42.50$

3b) $A = \frac{1}{2}(2)(\sqrt{15}) = \sqrt{15} \text{ ft}^2$
$\sqrt{15}(8.50) = \$32.92$

3c) $A = \frac{1}{2}(6)(3\sqrt{3}) = 9\sqrt{3} \text{ ft}^2$
$9\sqrt{3}(8.50) = \$132.50$

My Notes

6a) two rectangles and two triangles

6b) Both are correct. Creating a large rectangle and subtracting the areas gets the same area.

6c) 96 square units

6d) It is easier to determine the unknown side.

8) In the right triangle, the measure of one of the legs is the height of the triangle. In an equilateral triangle you can use the Pythagorean Theorem.

4. Reason abstractly. Rashid draws several tabletops in the shape of an equilateral triangle, each having different side lengths. Derive an area formula that will work for all equilateral triangles with side length *s*.

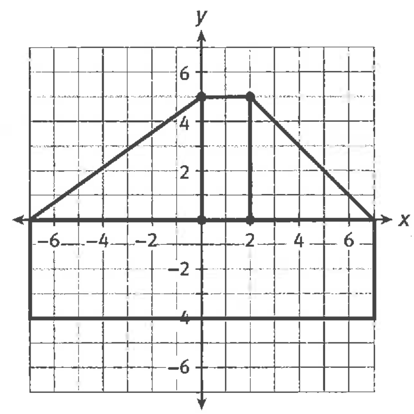
$\frac{1}{2}(s)(\frac{1}{2}s\sqrt{3}) = \frac{\sqrt{3}}{4}s^2$

5. Use the formula created in Item 4 to find the area of an equilateral triangle with side length 6 feet. Compare your answer to the area you found in Item 3c.

Area $= 9\sqrt{3}$ ft^2

Check Your Understanding

6. Find the area of the tabletop shown on the coordinate plane.



a. Name the shapes that make up the tabletop.

b. Critique the reasoning of others. Jalen and Bree each used a different method to determine the area of the tabletop. Jalen found the sum of the areas of each of the four figures. Bree drew auxiliary lines to create a large rectangle, and then subtracted the area of the two triangular regions not in the tabletop from the large rectangle. Whose method is correct? Explain your reasoning.

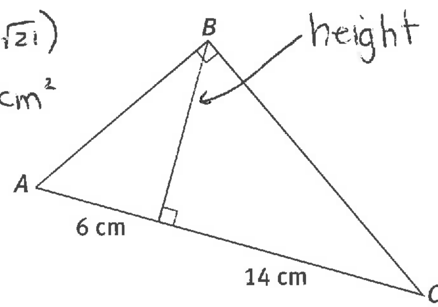
c. Compute the area of the tabletop.

d. What are some benefits of working with composite figures on a coordinate plane?

7. Determine the area of triangle *ABC*. Leave the answer in radical form.

$A = \frac{1}{2}(20)(2\sqrt{21})$

$A = 20\sqrt{21}$ cm^2



height $= \frac{6}{x} = \frac{x}{14}$

$x^2 = 84$

$x = 2\sqrt{21}$ cm

$b = 20$cm

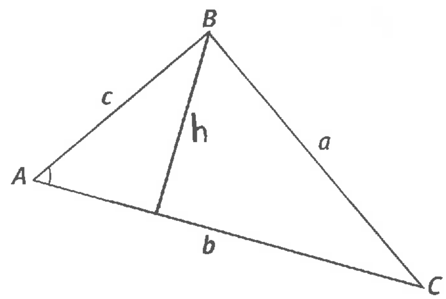
$h = 2\sqrt{21}$ cm

8. Explain how to determine the height of a right triangle versus the height of an equilateral triangle, given the side lengths of the triangles.

My Notes

Next, Rashid shows Lisa how to determine the area of a triangular-shaped tabletop given two side lengths and the measure of the included angle.

9. Make use of structure. Complete the proof to discover an area formula that can be used when given a triangle with two side lengths and the measure of the included angle.



a. Write the formula for the area of a triangle. Area $= \frac{1}{2}bh$

b. Draw and label a perpendicular line, h, from point B to $\overline{AC}$. This represents the height of the triangle.

c. Complete to find sin A: $\sin A = \dfrac{opposite}{hypotenuse} = \dfrac{h}{c}$

d. Solve for h: $(c)(\sin A) = h$

e. Substitute the equivalent expression you got for h (from part d) into the formula for area of a triangle (from part a).

Area $= \frac{1}{2}bh$

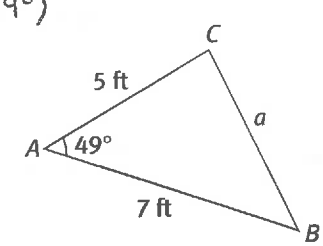
$= \frac{1}{2}b\,c(\sin A)$

10. The diagram shown represents tabletop ABC. The tabletop has side lengths 5 ft and 7 ft. The angle between the two sides, $\angle A$, has a measure of 49°. Compute the area of the triangular tabletop, to the nearest tenth.

Area $= \frac{1}{2}bc(\sin A)$

$A = \frac{1}{2}(7)(5)(\sin 49°)$

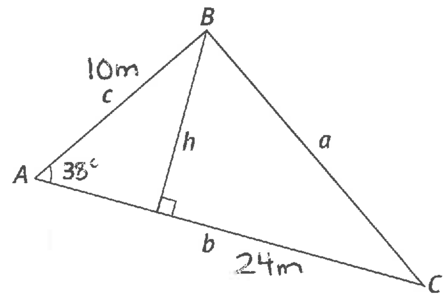
$A = 13.2$ ft^2



My Notes

Check Your Understanding

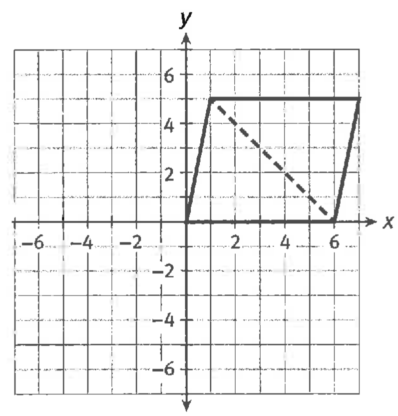
11. Find the area of the triangle to the nearest tenth, if $c = 10$ m, $b = 24$ m, and $m\angle A = 38°$.



$A = \frac{1}{2}(24)(10)(\sin 38°)$

$A = 73.9\text{m}^2$

12. A parallelogram is given on a coordinate plane. Use the area formula of a triangle to derive the area formula of a parallelogram. Then compute the area of the figure.



Area of one triangle:

$A = \frac{1}{2}bh$

Area of two triangles:

$A = 2(\frac{1}{2}bh)$

$A = bh$

LESSON 30-2 PRACTICE

For Items 13–16, compute the area of each triangular-shaped tabletop, to the nearest tenth.

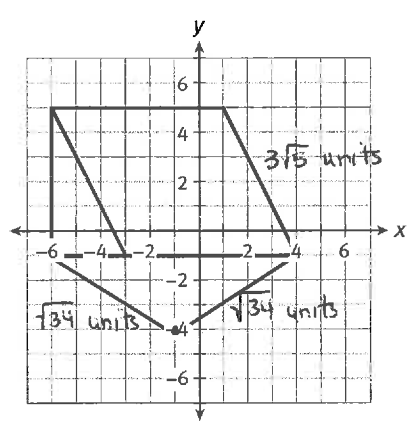
13. Equilateral triangle with 8.5 ft sides

14. Triangle *ABC* with vertices at (2, 1), (8, 1), and (6, 4)

15. Base measure of 15 ft, another side length of 12 ft, and an included angle of 30°

16. Isosceles triangle with sides of 4 ft, 7 ft, and 7 ft

17. **Make sense of problems.** Determine the perimeter and area of the tabletop shown on the coordinate plane below.

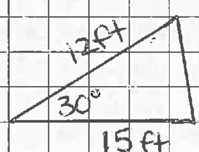


13) $A = \sqrt{12.75(12.75-8.5)(12.75-8.5)(12.75-8.5)}$

$A = 31.3\text{ft}^2$

14) $A = 9\text{ ft}^2$

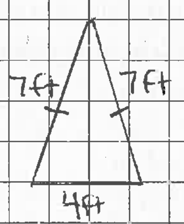
15)



$A = \frac{1}{2}(15)(12)(\sin 30°)$

$A = 45\text{ft}^2$

16)



$s = \frac{18}{2} = 9$

$A = \sqrt{9(9-7)(9-7)(9-4)}$

$A \approx 26.8\text{ft}^2$

17) $P = 31.4\text{ft}$

$A = 66\text{ft}^2$

My Notes

Area of trapezoid

$A = \frac{1}{2}(b_1 + b_2)(h)$

Area of rhombus
(and kite)

$A = \frac{1}{2}d_1d_2$

d_1 and d_2 are
the diagonals

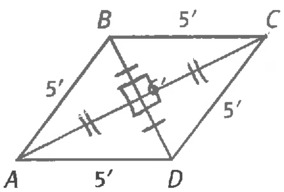
Learning Targets:

- Solve problems using the areas of rhombuses, trapezoids, and composite figures.
- Solve problems involving density.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Think-Pair-Share, Create Representations, Quickwrite

Lisa begins to explore other tabletop templates. She uses what she has learned about rectangles, triangles, and parallelograms to investigate the areas of other polygons.

1. Included in the tabletop templates is the rhombus shown.



a. List the properties of a rhombus that relate to the diagonals.

Diagonals are perpendicular bisectors of each other

b. Find the length of diagonal $\overline{AC}$.

8 feet

c. Apply the formula for the area of the triangles formed by the diagonals to find the area of the rhombus.

Area of each triangle $= \frac{1}{2}(4)(3) = 6\text{ ft}^2$

Total area $= 24\text{ ft}^2$

d. Derive a formula for the area of a rhombus with diagonal lengths d_1 and d_2.

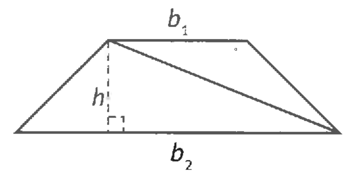
$4\left(\frac{1}{2}\left(\frac{1}{2}d_1\right)\left(\frac{1}{2}d_2\right)\right) = \frac{1}{2}d_1d_2$

My Notes

The area of a trapezoid with base lengths b_1 and b_2 and height h can be derived by applying what you have already learned about the area of a triangle.

2. Use the figure shown to derive a formula for the area of a trapezoid. Explain how you arrived at your answer.

$\frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}h(b_1+b_2)$

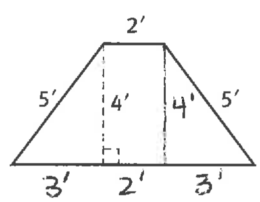


3. Critique the reasoning of others. Lisa states, "The area of a trapezoid is equal to the length of its median times its height." Is she correct? Why?

Yes; because the median $= \frac{1}{2}(b_1+b_2)$

4. Find the area of each of the trapezoids shown below.

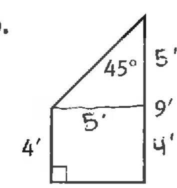
a.



$b_1 = 2\text{ft}$
$b_2 = 8\text{ft}$
$h = 4\text{ ft}$

$A = \frac{1}{2}(2+8)(4)$
$A = 20\text{ ft}^2$

b.



$b_1 = 4\text{ ft}$
$b_2 = 9\text{ ft}$
$h = 5\text{ft}$

$A = \frac{1}{2}(4+9)(5)$
$A = 32.5\text{ft}^2$

My Notes

MATH TERMS

Density is the mass per unit volume of a substance. You can determine the density of a substance using the following formula:

$$\text{density} = \frac{\text{mass}}{\text{volume}}.$$

To determine the total cost of a tabletop, Lisa needs to also consider the cost of shipping. The greater the mass of an object, the greater the shipping cost. The mass of the tabletop is dependent on the ***density*** of the material used in its construction.

5. A Cut Above is making the tabletop shown in Item 4a out of two different types of wood: western red cedar and maple. The volume of each tabletop is 4 ft^3. The density of western red cedar is 23 lb/ft^3, and the density of maple is 45 lb/ft^3. If the cost of shipping the tabletops is \$0.50 per pound, which tabletop costs more to ship? How much more?

Western red cedar: mass $= 23\,lb/ft^3 (4\,ft^3)$
$= 92\,lb$; \$46

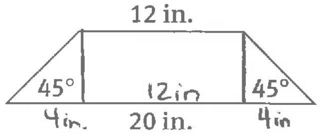
Maple: mass $= 45\,lb/ft^3(4\,ft^3) = 180\,lb$; \$90

The maple tabletop costs \$44 more to ship than it costs to ship the western red cedar tabletop

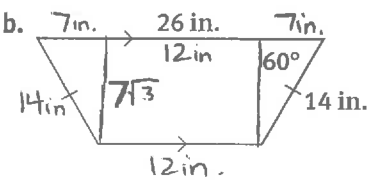
Check Your Understanding

6. Find the area of each trapezoid.

a.



b.



7. The density of bamboo is about 20 lb/ft^3. What is the mass of a bamboo tabletop that has a volume of 30 ft^3?

6a) $A = \frac{1}{2}(12+20)(4)$
$= 64\ in^2$

6b) $A = \frac{1}{2}(12+26)(7\sqrt{3})$
$= 230.4\ in^2$

7) 600 lb

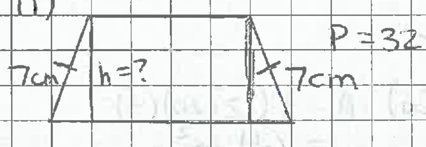
My Notes

8) $7\ ft^2$

9) $A = \frac{1}{2}(6)(10) = 30\ in^2$

10) $A = \frac{1}{2}(14+30)(8\sqrt{3})$
$= 176\sqrt{3}\ in^2$

11)



$54 = b_1 + b_2 + 7 + 7$
$b_1 + b_2 = 18cm$
$54 = \frac{1}{2}(18)(h)$
$54 = 9h$
$h = 6cm$

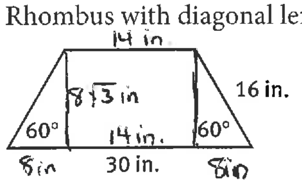
LESSON 30-3 PRACTICE

For Items 8–11, find the area, given the measures of each figure.

8. Isosceles trapezoid with base lengths 2 ft and 5 ft and leg lengths 2.5 ft

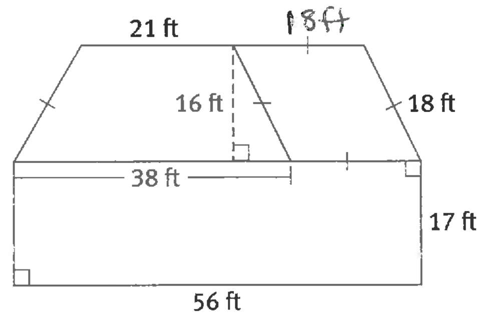
9. Rhombus with diagonal lengths 6 in. and 10 in.

10.



11. The area of an isosceles trapezoid is 54 square cm. The perimeter is 32 cm. If a leg is 7 cm long, find the height of the trapezoid.

12. Attend to precision. The front face of a manufactured home is shown in the diagram below.



a. Compute the area.

b. If the density of the material from which the home is made is 38 lb/ft^3, and the volume of the home is 856 ft^3, what is the mass?

Area of rectangle $= (56)(17) = 952 ft^2$
Area of trapezoid $= \frac{1}{2}(39+56)(16) = 760 ft^2$
Total area $= 1712 ft^2$

Mass $= 856(38) = 32,528$ lb.